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On the cycle expansion for the Lyapunov exponent of a product of random matrices

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Abstract
The cycle expansion of the thermodynamical zeta function for the Lyapunov exponent of a product of random matrices typically converges exponentially with the maximal cycle length (Mainieri 1992 *Phys. Rev. Lett.* **68** 1965). In this paper we show that the convergent exponents are given by the spectrum of a properly defined evolution operator, which describes how a steady distribution of vector direction is established under the action of random matrices. The exponential decay terms are automatically eliminated in the cycle expansion of the spectral determinant, which greatly accelerates the convergence provided all matrix elements are positive numbers. As a marginal case, the random Fibonacci series is studied in detail, and it is shown that this method is helpful.

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1. Introduction

Product of random matrices (PRM) arises naturally in the study of disorder systems such as random Ising chain and wave localization in one-dimensional random potential. The Lyapunov exponent of a PRM system, which characterizes the exponential divergence of the product matrices, is defined as

$$\gamma \equiv \lim_{n\to\infty} \frac{1}{n} \langle \log ||M_n \cdots M_2 M_1|| \rangle, \tag{1}$$

where M_i's are random matrices, $\|\cdot\|$ is the matrix norm and $\langle \cdots \rangle$ denotes the ensemble average. Although γ is defined as the averaged divergent rate, as a generalized law of large number, it was proved that the product of most (with unity probability) sequences of random matrices grow as $e^{n\gamma}$ when $n \to \infty$ [1]. The Lyapunov exponent is a statistical quantity of great physical significance. For example, the mean free energy of a random Ising chain is given by γ, while the localization length of wave in random medium is equal to γ^{-1} (for details, see [2] and references therein). For a general PRM, precise determination of γ is, however, not easy. The numerical performances of conventional methods such as Monte Carlo simulation and weak disorder expansion are seriously limited by slow convergence.

Inspired by the striking efficiency of periodic orbit theory in the study of statistical properties of chaotic dynamical systems [3, 4], Mainieri in 1992 proposed a cycle expansion of γ when quenched disorder is adopted. In this model, the random matrices are independently sampled from a discrete set with assigned probabilities, say, $M_i = A$ (or B) with probability p (or $q = 1 - p$). Starting from the thermodynamical zeta function, a cycle expansion of γ was derived [5],

$$\gamma = p\gamma_A + q\gamma_B + pq(\gamma_{AB} - \gamma_A - \gamma_B) + p^2 q(\gamma_{AAB} - \gamma_A - \gamma_{AB}) + pq^2(\gamma_{ABB} - \gamma_B - \gamma_{AB}) + \cdots, \tag{2}$$

where $\gamma_M \equiv \lim_{n\to\infty} \frac{1}{n} \log\|M^n\| = \log|\mu_0(M)|$ with $\mu_0(M)$ being the leading eigenvalue of M. The cycle expansion typically converges exponentially with the length of the longest cycles, which makes this method numerically superior to conventional approaches.

The efficiency of the cycle expansion of γ is largely controlled by the convergent exponent. However, the meaning of this exponent and how to determine it remain undiscussed. Moreover, from the viewpoint of periodic orbit theory, one can expect an even faster convergence if the spectral determinant instead of a thermodynamical zeta function is used. In fact, such method has been established in deterministic systems ([6], see also [7, appendix H]). The main idea is that we should consider the change in the direction of tangent vector, i.e. the induced dynamics on the unit sphere in the tangent space. In this paper we generalize this method to PRM, which is viewed as a stochastic dynamical system. As we can see, in the induced dynamical system, the convergence of the cycle expansion can be regarded as a process of relaxation, i.e. approaching an equilibrium state, and the convergent exponent can be related to the spectrum of the corresponding evolution (Frobenius-Perron) operator.

For the sake of simplicity, we focus on matrices with positive elements. The significance of this restriction will be clear in the discussion of the evolution operator. The paper is organized as follows. In section 2, we briefly describe the thermodynamical zeta function and show that the cycle expansion derived from it can be reduced to a more heuristic form, i.e. the average over periodic sequences. In section 3, the spectral determinant approach is established and its relation to the *replica* method is discussed. Numerical examples are given in section 4; in particular, the random Fibonacci series as a prototype of a degenerated system with intermittence is discussed in detail.

2. Thermodynamical zeta function

In this section we briefly discuss the thermodynamical zeta function formalism for the calculation of the Lyapunov exponent [5]. A description of this well-established result is necessary in order both to fix notation, and to provide a natural comparison with the evolution operator approach to be discussed in the following section.

Assume the random matrix M_i has m possible choices, namely $M_i = A_j$ with probability p_j, $j = 1, 2, \ldots, m$. With apparent reason, we require $p_j > 0$ and $\sum_{j=1}^{m} p_j = 1$. For convenience, the product of n matrices $A_{s_n} \cdots A_{s_2} A_{s_1}$ is denoted by A_S, where $S = s_n \cdots s_2 s_1$ is a string of integers. The probability of A_S is given by

$$\mathrm{Prob}(S) = \prod_{i=1}^{n} p_{s_i}. \tag{3}$$

In these notations, the Lyapunov exponent is explicitly defined as $\gamma = \lim_{n\to\infty} \gamma^{(n)}$ with

$$\gamma^{(n)} \equiv \frac{1}{n} \sum_{|S|=n} \text{Prob}(S) \log \mu_0(A_S), \tag{4}$$

where $|S|$ denotes the string length. In (4) we have fixed $\|A\|$ to $\mu_0(A)$ for simplicity.

To calculate γ and understand the multi-scale growth of the matrix products, it is convenient to start with the following auxiliary summation:

$$\sum_{|S|=n} \text{Prob}(S)\mu_0^\beta(A_S) \tag{5}$$

which diverges as $e^{n\lambda_\beta}$ when $n \to \infty$. λ_β is called the *generalized Lyapunov exponent*, from which γ is determined as

$$\gamma = \partial_\beta \lambda_\beta|_{\beta=0}. \tag{6}$$

When β is an integer, λ_β can be related to the leading eigenvalue of a finite matrix. Based on this fact, the *replica* method provides an estimation of γ via the extrapolation formula $\gamma \approx 2\lambda_1 - \lambda_2/2$ [2]. Generally, we have

$$\lambda_\beta = -\log z_\beta, \tag{7}$$

where z_β is the smallest zero of the thermodynamical zeta function defined by[1]

$$\zeta^{-1}(z,\beta) = \exp\left\{-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{|S|=n} \text{Prob}(S)\mu_0^\beta(A_S)\right\}. \tag{8}$$

Obviously, $\zeta^{-1}(z,0) = 1 - z$ and $z_0 = 1$. Therefore the Lyapunov exponent can be calculated from the thermodynamical zeta function according to

$$\gamma = -\partial_\beta z_\beta|_{\beta=0} = \left.\frac{\partial_\beta \zeta^{-1}(z,\beta)}{\partial_z \zeta^{-1}(z,\beta)}\right|_{\beta=0,z=1} = -\partial_\beta \zeta^{-1}(z,\beta)|_{\beta=0,z=1}. \tag{9}$$

Since $\mu_0(A_S)$ as well as Prob(S) are invariant under a cyclic permutation of S, i.e. $S = s_n \cdots s_2 s_1 \to S' = s_1 s_n \cdots s_2$, the summation in (8) can be performed in terms of cycles and their repetitions, namely

$$\begin{aligned}\zeta^{-1}(z,\beta) &= \exp\left\{-\sum_{\{\text{p}\}} \sum_{r=1}^{\infty} \frac{1}{r} [z^{n_\text{p}} \text{Prob}(\text{p})\, e^{\beta\gamma_\text{p}}]^r\right\} \\ &= \prod_{\{\text{p}\}} [1 - z^{n_\text{p}} \text{Prob}(\text{p})\, e^{\beta\gamma_\text{p}}].\end{aligned} \tag{10}$$

In product (10) 'p' runs over all primitive cycles, n_p denotes the cycle length and $\gamma_\text{p} \equiv \log \mu_0(A_\text{p})$. Expanding $\zeta^{-1}(z,\beta)$ in powers of z, i.e.

$$\begin{aligned}\zeta^{-1}(z,\beta) &= \prod_i (1 - z p_i\, e^{\beta\gamma_i}) \prod_{i<j} (1 - z^2 p_i p_j\, e^{\beta\gamma_{ij}}) \cdots \\ &= 1 - z \sum_i p_i\, e^{\beta\gamma_i} - z^2 \sum_{i<j} p_i p_j (e^{\beta\gamma_{ij}} - e^{\beta(\gamma_i+\gamma_j)}) + \cdots.\end{aligned} \tag{11}$$

We obtain, according to (9), the cycle expansion of the Lyapunov exponent [5],

$$\gamma = \sum_i p_i \gamma_i + \sum_{i<j} p_i p_j (\gamma_{ij} - \gamma_i - \gamma_j) + \cdots. \tag{12}$$

[1] Throughout this paper, a zero is the smallest means that it is the zero with the smallest module.

In the cycle expansion, the main contribution is given by the short (fundamental) cycles, while the high-order corrections due to the longer cycles are naturally organized into nearly cancelled groups.

It should be noted that although the cycle expansion (12) appears physically appealing, it can be reduced to a more compact, and in a sense trivial, form. Combining (9) and (8), we have

$$\begin{aligned}\gamma &= \lim_{z\to 1} \zeta^{-1}(z,0) \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{|S|=n} \mathrm{Prob}(S) \log \mu_0(A_S) \\ &= \lim_{z\to 1}(1-z) \sum_{n=1}^{\infty} \gamma^{(n)} z^n = \lim_{z\to 1} \left[\gamma^{(1)} z + \sum_{n=2}^{\infty} (\gamma^{(n)} - \gamma^{(n-1)}) z^n \right]. \end{aligned} \tag{13}$$

Hence if the cycle expansion (12) is truncated based on the cycle length, it will reproduce definition (4). In fact, we can directly write $\gamma^{(n)}$ in terms of cycles, namely,

$$\gamma^{(n)} = \sum_{\{\mathrm{p}\}} \sum_{r=1}^{\infty} \delta_{n, r n_{\mathrm{p}}} \mathrm{Prob}(\mathrm{p})^r \gamma_{\mathrm{p}}, \tag{14}$$

e.g.

$$\gamma^{(1)} = \sum_i p_i \gamma_i \tag{15}$$

$$\gamma^{(2)} = \sum_i p_i^2 \gamma_i + \sum_{i<j} p_i p_j \gamma_{ij} \tag{16}$$

and so on. Evidently (14) and the truncations of (12) are equivalent.

3. Evolution operator approach

By using the thermodynamical method, it was observed that $\gamma^{(n)} \sim \gamma + c\delta^n$ when $n \to \infty$ [5]. Since the dynamical zeta function in the periodic orbit theory can be regarded as the first-order approximation of spectral determinant [7], it is natural to adopt the spectral determinant to improve the cycle expansion of the Lyapunov exponent because it allows faster than the exponential convergence. For this purpose, one must first associate the Lyapunov exponent to the spectrum of an evolution (or transfer) operator supported by a dynamical system. Such system has been successfully constructed in the deterministic case [6]. The main idea is that we should consider the change in the direction of tangent vector. By extending this idea to PRM, in this section we discuss in detail the evolution operator approach for the calculation of the Lyapunov exponent. We would like to point out that although the derivation is formal and a little bit lengthy, the final result is rather simple: we need only to replace $\gamma^{(n)}$ by a weighted averaging, namely

$$\overline{\gamma}_n = \frac{\sum_{j=0}^{n-1} a_j \gamma^{(n-j)}}{\sum_{j=0}^{n-1} a_j}, \tag{17}$$

where the coefficients are derived from $\mathcal{L}_0$, the evolution operator, according to

$$\frac{1}{1-z} \det(1 - z\mathcal{L}_0) = \sum_{k=0}^{\infty} a_k z^k, \tag{18}$$

and numerical results show that equation (17) is actually a very effective accelerated algorithm (see section 4).

3.1. Map on the unit vectors

We first consider 2×2 matrices for simplicity. Let X_θ be the unit vector in the direction labelled by θ. Write the action of a matrix A on X_θ as

$$AX_\theta = h_A(\theta) X_{f_A(\theta)}, \tag{19}$$

namely f_A describes the change in the vector direction and h_A for the change in the vector length. As $a_{ij} > 0$, we can consider only vectors with positive coordinates. To be concrete, let us define $\theta = \log(x_2/x_1)$ and $r = (x_1 x_2)^{1/2}$, then $X_\theta = (\mathrm{e}^{-\theta/2}, \mathrm{e}^{\theta/2})^T$,

$$f_A(\theta) = \log \frac{a_{21} + a_{22}\,\mathrm{e}^\theta}{a_{11} + a_{12}\,\mathrm{e}^\theta} \tag{20}$$

and

$$h_A(\theta) = (a_{11}a_{22} + a_{12}a_{21} + a_{11}a_{21}\,\mathrm{e}^{-\theta} + a_{12}a_{22}\,\mathrm{e}^\theta)^{\frac{1}{2}} = \left|\frac{\det A}{\partial_\theta f_A(\theta)}\right|^{\frac{1}{2}}. \tag{21}$$

The last equation indicates that the dilation in the r-direction can be extracted from the contraction in θ-space. This fact enables us to get the Lyapunov exponent from the dynamics of unit vectors. Moreover, we can see that f_A is actually a uniformly contractive map. Note that

$$|\partial_\theta f_A(\theta)| \leqslant \frac{|a_{11}a_{22} - a_{12}a_{21}|}{a_{11}a_{22} + a_{12}a_{21} + 2(a_{11}a_{21}a_{12}a_{22})^{\frac{1}{2}}} \equiv \kappa < 1, \tag{22}$$

and therefore

$$|f_A(\theta_1) - f_A(\theta_2)| \leqslant \kappa|\theta_1 - \theta_2| \tag{23}$$

for arbitrary θ_1, θ_2. The *uniform contraction property* is crucial to the evolution operator to be discussed. With an elaborate definition of projective distance, this property can be established for high-dimensional positive matrices (see, for example, [18]).

As a contractive map, f_A has a unique fixed point θ^*. X_{θ^*} is the eigenvector corresponding to the leading eigenvalue of A, i.e. $h_A(\theta^*) = \mu_0(A)$ and, according to equation (21),

$$\partial_\theta f_A(\theta^*) = \frac{\det A}{\mu_0^2(A)} = \frac{\mu_1(A)}{\mu_0(A)} \equiv g(A), \tag{24}$$

where $\mu_1(A)$ denotes the second eigenvalue of A.

3.2. Evolution operator

Let $\rho(\theta)$ be a distribution of the vector direction, its evolution under the action of random matrices $\{A_i\}$ is described by the operator $\mathcal{L}_0$, i.e.

$$(\mathcal{L}_0\rho)(\theta) = \int \mathcal{L}_0(\theta, \theta')\rho(\theta')\,\mathrm{d}\theta', \tag{25}$$

where

$$\mathcal{L}_0(\theta_1, \theta_2) = \sum_{i=1}^m p_i \delta[\theta_1 - f_{A_i}(\theta_2)]. \tag{26}$$

To get the Lyapunov exponent, it is necessary to introduce a generalized evolution operator

$$\mathcal{L}_\beta(\theta_1, \theta_2) = \sum_{i=1}^m p_i \left|\frac{\det A_i}{\partial_\theta f_{A_i}(\theta_2)}\right|^{\frac{\beta}{2}} \delta\big[\theta_1 - f_{A_i}(\theta_2)\big]. \tag{27}$$

The generalized evolution operator is close related to summation (5). Let us calculate the traces of $\mathcal{L}_\beta$ and its powers. If $t \geqslant 1$ is an integer

$$\mathcal{L}_\beta^t(\theta_1, \theta_2) = \sum_{|S|=t} \mathrm{Prob}(S) \left| \frac{\det A_S}{\partial_\theta f_{A_S}(\theta_2)} \right|^{\frac{\beta}{2}} \delta[\theta_1 - f_{A_S}(\theta_2)], \tag{28}$$

hence

$$\mathrm{tr}\left(\mathcal{L}_\beta^t\right) = \int_{-\infty}^{\infty} \mathcal{L}_\beta^t(\theta, \theta)\,\mathrm{d}\theta = \sum_{|S|=t} \frac{\mathrm{Prob}(S)\mu_0^\beta(A_S)}{1 - g(A_S)}. \tag{29}$$

Note that, when $t \to \infty$, the denominators will not affect the divergence behaviour of (29) if

$$\sup_S\{|g(A_S)|\} < 1. \tag{30}$$

In fact, from the uniform contraction property of f_{A_i} we have a more strong fact, namely the *hyperbolic condition*

$$|g(A_S)| < \eta^{|S|} \qquad (0 < \eta < 1). \tag{31}$$

Therefore we conclude that $\mathrm{tr}\left(\mathcal{L}_\beta^t\right)$ also diverges as $\mathrm{e}^{t\lambda_\beta}$ when $t \to \infty$, or, $\mathrm{e}^{\lambda_\beta}$ is the leading eigenvalue of $\mathcal{L}_\beta$.

The eigenvalues of $\mathcal{L}_\beta$ can be calculated from the zeros of the spectral determinant

$$\begin{aligned}\det(1 - z\mathcal{L}_\beta) &= \exp\left\{-\sum_{t=1}^{\infty} \frac{z^t}{t}\,\mathrm{tr}\left(\mathcal{L}_\beta^t\right)\right\} = \exp\left\{-\sum_S \frac{z^{|S|}}{|S|} \frac{\mathrm{Prob}(S)\mu_0^\beta(A_S)}{1 - g(A_S)}\right\} \\ &= \exp\left\{-\sum_{\{\mathrm{p}\}} \sum_{r=1}^{\infty} \frac{1}{r} \frac{\left[z^{n_\mathrm{p}}\mathrm{Prob}(\mathrm{p})\mu_0^\beta(A_\mathrm{p})\right]^r}{1 - g^r(A_\mathrm{p})}\right\} \\ &= \prod_{k=0}^{\infty} \zeta_k^{-1}(z, \beta),\end{aligned} \tag{32}$$

where

$$\begin{aligned}\zeta_k^{-1}(z, \beta) &= \exp\left\{-\sum_{\{\mathrm{p}\}} \sum_{r=1}^{\infty} \frac{1}{r}\left[z^{n_\mathrm{p}}\mathrm{Prob}(\mathrm{p})\mu_0^\beta(A_\mathrm{p})g^k(A_\mathrm{p})\right]^r\right\} \\ &= \prod_{\{\mathrm{p}\}}\left[1 - z^{n_\mathrm{p}}\mathrm{Prob}(\mathrm{p})\mu_0^\beta(A_S)g^k(A_\mathrm{p})\right].\end{aligned} \tag{33}$$

Obviously we can identify $\zeta_0^{-1}(z, \beta)$ with the thermodynamical zeta function. Although $\det(1 - z\mathcal{L}_\beta)$ and $\zeta_0^{-1}(z, \beta)$ have the same smallest zero, it is expected that the former is an entire analytic function of z while the latter generally has poles[2]. The analytic property of $\det(1 - z\mathcal{L}_\beta)$ ensures the power series expansion

$$\det(1 - z\mathcal{L}_\beta) = 1 + \sum_{k=1}^{\infty} b_k(\beta) z^k \tag{34}$$

converges no matter how large $|z|$ may be. Therefore, if the spectral determinant is truncated as

$$F_n(z, \beta) = 1 + \sum_{k=1}^{n} b_k(\beta) z^k, \tag{35}$$

[2] Although our numerical results strongly suggest that $\det(1 - z\mathcal{L}_\beta)$ is an entire function, a proof of this fact is mathematically very hard. We note also that, under certain conditions, the entireness of the spectral determinant can be rigorously proved (see, for example, [8]).

the error will vanish faster than any power of e^{-n} when $n \to \infty$. Calculating γ at this approximation, we get

$$\gamma \approx \frac{\partial_\beta F_n(z,\beta)}{\partial_z F_n(z,\beta)}\bigg|_{\beta=0,z=1} = \frac{-q_n + \sum_{j=0}^{n-1} a_j \gamma^{(n-j)}}{-na_n + \sum_{j=0}^{n-1} a_j}, \tag{36}$$

where a_n and q_n are determined by

$$H(z) = \frac{1}{1-z}\det(1 - z\mathcal{L}_0) = \exp\left\{-\sum_{\{p\}}\sum_{r=1}^{\infty}\frac{1}{r}\frac{[z^{n_p}\mathrm{Prob}(p)g(A_p)]^r}{1-g^r(A_p)}\right\}$$
$$= \prod_{k>0}\zeta_k^{-1}(z,0) = \sum_{k=0}^{\infty} a_k z^k \tag{37}$$

and

$$Q(z) = \partial_\beta \prod_{k>0}\zeta_k^{-1}(z,\beta)\bigg|_{\beta=0} = \sum_{k=1}^{\infty} q_k z^k. \tag{38}$$

Since $H(z)$ and $Q(z)$ are also entire functions, both a_n and q_n can be dropped out in (36) and we obtain the final form (17).

Heuristically, the super-exponential convergence of $\overline{\gamma}_n$ can be understood as follows. Write

$$\gamma^{(n)} = \gamma + \sum_\alpha c_\alpha \nu_\alpha^n + R_n, \tag{39}$$

where $|\nu_\alpha| < 1$ and R_n is a super-exponentially small remainder. Thus

$$\overline{\gamma}_n = \gamma + \frac{\sum_{j=0}^{n-1} a_j R_{n-j} + \sum_\alpha \left[H\left(\nu_\alpha^{-1}\right) - H_n\left(\nu_\alpha^{-1}\right)\right] c_\alpha \nu_\alpha^n}{H(1) - H_n(1)}, \tag{40}$$

where $H_n(z) = \sum_{j=n}^{\infty} a_j z^j$. Accordingly, the super-exponential convergence of $\overline{\gamma}_n$ implies $H\left(\nu_\alpha^{-1}\right) = 0$, or ν_α's belong to the spectrum of $\mathcal{L}_0$. In other words, $\gamma^{(n)} \to \gamma$ can be viewed as a relaxation process described by the evolution operator and its exponentially decaying terms are automatically eliminated in the weighted average (17).

3.3. High-dimensional matrices and replica tricks

We can readily generalize the evolution operator method to $d \times d$ matrices. The trace of the generalized evolution operator or its power is given by

$$\mathrm{tr}\left(\mathcal{L}_\beta^t\right) = \sum_{|S|=t} \frac{\mathrm{Prob}(S)\mu_0^\beta(A_S)}{\prod_{j=1}^{d-1}[1 - g_j(A_S)]}, \tag{41}$$

where $g_j(A) = \mu_j(A)/\mu_0(A)$ with $\{\mu_0(A), \mu_1(A)\cdots\mu_{d-1}(A)\}$ being the spectrum of A. The spectral determinant reads

$$\det(1 - z\mathcal{L}_\beta) = \exp\left\{-\sum_{\{p\}}\sum_{r=1}^{\infty}\frac{1}{r}\frac{\left[z^{n_p}\mathrm{Prob}(p)\mu_0^\beta(A_p)\right]^r}{\prod_{j=1}^{d-1}\left[1 - g_j^r(A_p)\right]}\right\}$$
$$= \prod_{\mathbf{k}} \zeta_{\mathbf{k}}^{-1}(z,\beta), \tag{42}$$

where $\mathbf{k} = (k_1, k_2, \ldots, k_{d-1})$, $k_j \geqslant 0$, and

$$\zeta_{\mathbf{k}}^{-1}(z,\beta) = \prod_{\{p\}}\left[1 - z^{n_p}\mathrm{Prob}(p)\mu_0^\beta(A_p)\Pi_{j=1}^{d-1} g_j^{k_j}(A_p)\right]. \tag{43}$$

Then we consider the generalized Lyapunov exponent at integer indices. It is clear that λ_n can be extracted from a $d^n \times d^n$ matrix based on the following observation [2]:

$$[\mathrm{tr}(A_S)]^n = \mathrm{tr}\left(\bigotimes^n A_S\right) = \mathrm{tr}\left(\bigotimes^n A_{s_n} \cdots \bigotimes^n A_{s_2} \bigotimes^n A_{s_1}\right), \tag{44}$$

where $\bigotimes^n A$ denotes n times tensor product of A. So

$$\sum_{|S|=t} \mathrm{Prob}(S)[\mathrm{tr}(A_S)]^n = \mathrm{tr}\left\{\left[\sum_{i=1}^{m} p_i \bigotimes^n A_i\right]^t\right\} \equiv \mathrm{tr}\left(\mathcal{D}_n^t\right). \tag{45}$$

Note that $\mathrm{tr}(A_S)$ is dominated by $\mu_0(A_S)$ and, similar to (5), summation (45) diverges also as $\mathrm{e}^{t\lambda_n}$ when $t \to \infty$. Consequently, we find that e^{λ_n} is given by the leading eigenvalue of $\mathcal{D}_n$. It is then natural to ask how $\det(1 - z\mathcal{D}_n)$ can be related to $\det(1 - z\mathcal{L}_n)$. The answer is

$$\prod_{|\mathbf{k}|\leqslant n} \zeta_{\mathbf{k}}^{-1}(z, n) = \det(1 - z\mathcal{M}_n), \tag{46}$$

where $|\mathbf{k}| = \sum_{j=1}^{d-1} k_j$ and $\mathcal{M}_n$ is the restriction of $\mathcal{D}_n$ within the invariant subspace consists of symmetric tensors. In other words, $\det(1 - z\mathcal{L}_n)$ and $\det(1 - z\mathcal{D}_n)$ have a common factor $\det(1 - z\mathcal{M}_n)$, by which the shared leading eigenvalue of $\mathcal{L}_n$ and $\mathcal{D}_n$ can be determined. The dimensionality of $\mathcal{M}_n$ is $(n+d-1)!/n!(d-1)!$, which is much smaller than that of $\mathcal{D}_n$ when n is large. The proof of (46) is straightforward. Note that, within the symmetric invariant subspace, the trace of a tensor product $\bigotimes^n A$ is given by

$$\mathrm{tr}\left(\bigotimes^n A\right)\Big|_0 = \sum_{|\mathbf{k}'|=n} \Pi_{j=0}^{d-1} \mu_j^{k_j}(A), \tag{47}$$

where $\mathbf{k}' = (k_0, k_1, \ldots, k_{d-1})$ and k_j are non-negative integers. Therefore we have

$$\begin{aligned}
\prod_{|\mathbf{k}|\leqslant n} \zeta_{\mathbf{k}}^{-1}(z, n) &= \exp\left\{-\sum_S \frac{z^{|S|}}{|S|}\mathrm{Prob}(S)\mu_0^n(A_S) \sum_{|\mathbf{k}|\leqslant n} \Pi_{j=1}^{d-1} g_j^{k_j}(A_S)\right\} \\
&= \exp\left\{-\sum_S \frac{z^{|S|}}{|S|}\mathrm{Prob}(S) \sum_{|\mathbf{k}|\leqslant n} \mu_0^{n-|\mathbf{k}|}(A_S)\Pi_{j=1}^{d-1}\mu_j^{k_j}(A_S)\right\} \\
&= \exp\left\{-\sum_{t=1}^{\infty} \frac{z^t}{t} \sum_{|S|=t} \mathrm{Prob}(S) \sum_{|\mathbf{k}'|=n} \Pi_{j=0}^{d-1}\mu_j^{k_j}(A_S)\right\} \\
&= \exp\left\{-\sum_{t=1}^{\infty} \frac{z^t}{t} \mathrm{tr}(\mathcal{D}^t)\Big|_0\right\} = \exp\left\{-\sum_{t=1}^{\infty} \frac{z^t}{t} \mathrm{tr}\left(\mathcal{M}_n^t\right)\right\} \\
&= \det(1 - z\mathcal{M}_n).
\end{aligned} \tag{48}$$

4. Numerical examples

Our first example has its origin in the study of the Ising chain with a random magnetic field [2, 5], i.e.

$$A_1 = \begin{bmatrix} 1 & \mathrm{e}^{-2h-2J} \\ \mathrm{e}^{-2J} & \mathrm{e}^{-2h} \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & \mathrm{e}^{2h-2J} \\ \mathrm{e}^{-2J} & \mathrm{e}^{2h} \end{bmatrix} \tag{49}$$

and $p_1 = p_2 = 1/2$.

Before discussing the numerical result, it is advisable to take this example to give a brief description of the computational aspects of the cycle expansion method (for more details, please consult [7]). Note first that, for a 2×2 matrix A,

$$\mu_0(A) = \tfrac{1}{2}[\mathrm{tr}(A) + \sqrt{\mathrm{tr}^2(A) - 4\det(A)}] \qquad \text{and} \qquad g(A) = \frac{\mathrm{tr}(A)}{\mu_0(A)} - 1.$$

Then, from equation (14), $\gamma^{(n)}$ can be calculated, e.g.

$$\begin{aligned}
\gamma^{(1)} &= \tfrac{1}{2}[\log\mu_0(A_1) + \log\mu_0(A_1)] \equiv \tfrac{1}{2}(\gamma_1 + \gamma_2)\\
\gamma^{(2)} &= \tfrac{1}{4}[\log\mu_0(A_1) + \log\mu_0(A_1) + \log\mu_0(A_1A_2)] \equiv \tfrac{1}{4}(\gamma_1 + \gamma_2 + \gamma_{12})\\
\gamma^{(3)} &= \tfrac{1}{8}(\gamma_1 + \gamma_2 + \gamma_{112} + \gamma_{122})\\
\gamma^{(4)} &= \tfrac{1}{16}(\gamma_1 + \gamma_2 + \gamma_{12} + \gamma_{1112} + \gamma_{1222} + \gamma_{1122})
\end{aligned}$$

and so on. To get the averaging weights $\{a_k\}$, it is convenient to introduce an auxiliary series $\{c_n\}$ according to

$$H(z) = \sum_{k=0}^{\infty} a_k z^k = \exp\left(-\sum_{n=1}^{\infty} \frac{c_n}{n} z^n\right),$$

from which a_k can be recursively extracted. Namely, $a_0 = 1$ and

$$a_k = -\frac{1}{k}\sum_{j=1}^{k} c_j a_{k-j}$$

for $k > 0$. c_n is determined by equation (37),

$$c_n = \mathrm{tr}\,\mathcal{L}_0^n - 1 = \sum_{\{\mathrm{p}\}}\sum_{r=1}^{\infty} \delta_{n,rn_\mathrm{p}} n_p \frac{[\mathrm{Prob}(\mathrm{p})g(A_\mathrm{p})]^r}{1 - g^r(A_\mathrm{p})}$$

e.g.

$$\begin{aligned}
c_1 &= \frac{1}{2}\left[\frac{g(A_1)}{1 - g(A_1)} + \frac{g(A_2)}{1 - g(A_2)}\right]\\
c_2 &= \frac{1}{4}\left[\frac{g^2(A_1)}{1 - g^2(A_1)} + \frac{g^2(A_2)}{1 - g^2(A_2)} + \frac{2g(A_{12})}{1 - g(A_{12})}\right]\\
c_3 &= \frac{1}{8}\left[\frac{g^3(A_1)}{1 - g^3(A_1)} + \frac{g^3(A_2)}{1 - g^3(A_2)} + \frac{3g(A_{112})}{1 - g(A_{112})} + \frac{3g(A_{122})}{1 - g(A_{122})}\right]\\
c_4 &= \frac{1}{16}\left[\frac{g^4(A_1)}{1 - g^4(A_1)} + \frac{g^4(A_2)}{1 - g^4(A_2)} + \frac{2g^2(A_{12})}{1 - g^2(A_{12})} + \frac{4g(A_{1112})}{1 - g(A_{1112})} + \frac{4g(A_{1222})}{1 - g(A_{1222})}\right.\\
&\qquad \left. + \frac{4g(A_{1122})}{1 - g(A_{1122})}\right]
\end{aligned}$$

and so on. (Note that $A_{12} \equiv A_1A_2$, $A_{112} \equiv A_1A_1A_2$, etc.) Finally, we obtain $\overline{\gamma}_n$ according to equation (17).

The numerical result at $J = 0.3$ and $h = 1.4$ is shown in figure 1. Evidently $\overline{\gamma}_n$ converges much faster than $\gamma^{(n)}$. For example, $\overline{\gamma}_{11}$ produces more than 40 reliable digits

$$\gamma = 1.177\,274\,472\,212\,277\,140\,131\,895\,216\,552\,086\,922\,6976\cdots,$$

while $\gamma^{(11)}$ gives only 12 correct digits. Moreover, we can see that the convergence of $\gamma^{(n)}$ is dominated by the leading zero of $H(z)$ as expected.

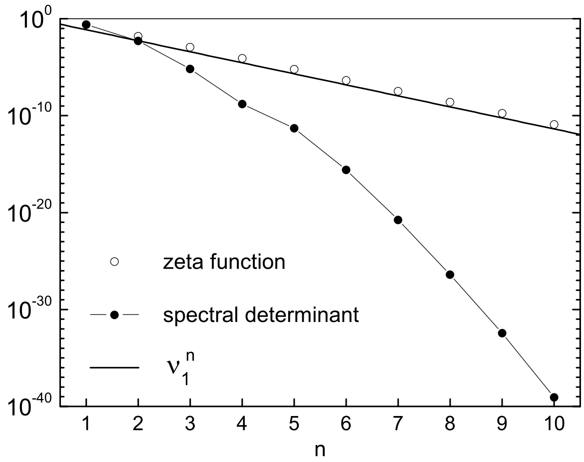

Figure 1. The plot of $|\gamma^{(n)} - \gamma|$ and $|\overline{\gamma}_n - \gamma|$ versus n for system (49). The solid line shows the exponential decay of ν_1^n, where $1/\nu_1 \approx 13.724\,542\,604\,9653$ is the smallest zero of $H(z)$.

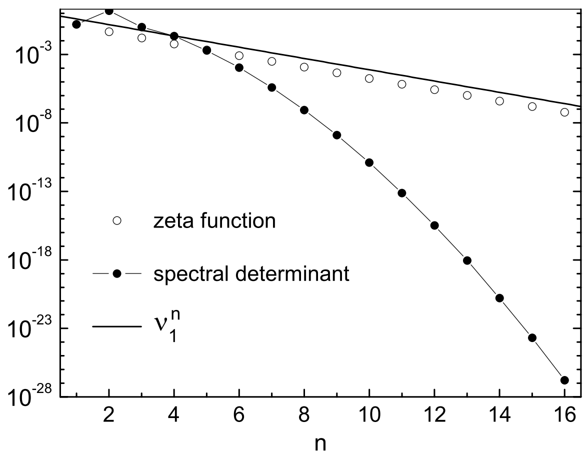

Figure 2. As figure 1, for system (50). $\gamma \approx 0.433\,724\,901\,564\,827\,884\,296\,173\,01$ and $1/\nu_1 \approx 2.582\,086\,291\,210\,538$.

Then we consider

$$A_1 = \begin{bmatrix} 1 & 1 \\ \epsilon & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & \epsilon \\ 1 & 1 \end{bmatrix} \tag{50}$$

and $p_1 = p_2 = 1/2$. The degenerated case at $\epsilon = 0$ appears in the study of the Farey fraction spin chain [9]. We calculate a nearly degenerated system at $\epsilon = 0.1$ (see figure 2). The qualitative convergent behaviour of $\overline{\gamma}_n$ and $\gamma^{(n)}$ remain the same as that in the first example. Note that, due to the nearby degeneracy, the precision of $\overline{\gamma}_n$ is even poorer than $\gamma^{(n)}$ when n is small.

Finally we consider the random Fibonacci sequence defined via the recurrence $x_{n+1} = x_n \pm x_{n-1}$, where the signs are chosen independently with equal probabilities. Several works have been done to this system (see, for example, [10–13]). Based on a fractal measure on the Stern–Brocot division of the real line, Viswanath showed that

$$e^{\gamma_f} = \lim_{n\to\infty} |x_n|^{1/n} \approx 1.131\,988\,24\cdots$$

with unity probability [10]. Obviously γ_f is the Lyapunov exponent of the PRM defined by

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \tag{51}$$

and $p_1 = p_2 = 1/2$. The presence of the negative matrix element makes this system not a suitable starting point to calculate γ_f. In order to proceed, we can make use of the transformation that maps the system to the product of two non-negative matrices

$$A_1 = A \equiv \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad A_2 = B \equiv \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \tag{52}$$

with $p_1 = \tau$ and $p_2 = \tau^2$, where $\tau = (\sqrt{5}-1)/2$ is the golden ratio. γ_f can be calculated from the new Lyapunov exponent γ according to $\gamma_f = \frac{\tau}{2}\gamma$ [10]. We cannot straightforwardly apply the evolution operator method to the non-negative system (52) since $A_{12} = A_1 A_2$ is degenerated, i.e. $g(A_{12}) = 1$. Just as what happened in intermittent dynamical systems, the thermodynamical zeta function $\zeta_0^{-1}(z, \beta)$ has a branch point [14, 15]. However, since the statistical weight appears as an effective stable exponent, the branch point is located at $z_c = -\tau^{-3/2}$ rather than 1 for the dynamical zeta function in the deterministic system. The branch point imposes a lower bound $n^\alpha |z_c|^{-n}$ to the convergence of the cycle expansion.

Two strategies have been proposed to cope with the singularity caused by intermittence. The first is to truncate the cycle expansion of the dynamical zeta function based on cycle stability rather than cycle length [16]. The second is to generalize the cycle expansion of $\zeta_0^{-1}(z, \beta)$ around the branch point based on its analytical structure [17]. In our case, without essential modification of the cycle expansion method, reasonable precision can be achieved by making use of two simple techniques. The first is renewal of matrices. Specifically, we redefine a model of PRM with $m = 2l + 1$ fundamental matrices

$$A_1 = (BA)^l, \qquad A_{2j} = B(BA)^{j-1} \qquad \text{and} \qquad A_{2j+1} = A^2(BA)^{j-1} \tag{53}$$

$j = 1, 2, \ldots, l$. The corresponding probabilities are inherited from the original system, i.e. $p_1 = \tau^{3l}$ and $p_{2j} = p_{2j+1} = \tau^{3j-1}$. We can get γ_f from the new Lyapunov exponent γ according to

$$\frac{\gamma_f}{\gamma} = \frac{\tau}{2}\left[2l\tau^{3l} + \sum_{j=1}^{l}(4j-1)\tau^{3j-1}\right]^{-1}. \tag{54}$$

In this way the branch point is moved to $z_c = \tau^{-3l}$ hence $\gamma^{(n)}$ will converge much faster than that in the original system (see figure 3). The second is to truncate the spectral determinant as the product of the finite number of zeta functions, i.e., to approximate $\det(1 - z\mathcal{L}_0)$ by

$$F_k(z) = \prod_{j=0}^{k} \zeta_j^{-1}(z, 0) = (1-z)\sum_{j=0}^{\infty} a_{k,j} z^j \tag{55}$$

and calculate γ by the weighted averaging

$$\gamma \approx \overline{\gamma}_{k,n} = \frac{\sum_{j=0}^{n-1} a_{k,j}\gamma^{(n-j)}}{\sum_{j=0}^{n-1} a_{k,j}}. \tag{56}$$

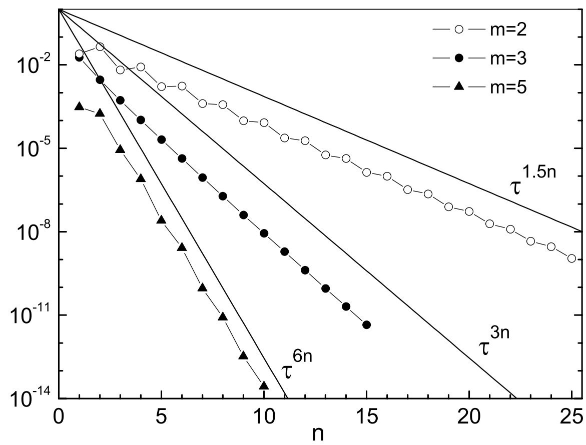

Figure 3. Convergence of the cycle expansion of the thermodynamical zeta function with different choices of fundamental matrices (see equations (53) and (54)). $|\gamma_f^{(n)} - \gamma_f|$ is plotted versus the maximal cycle length n. $m = 2$ corresponds to the original system given by equation (52).

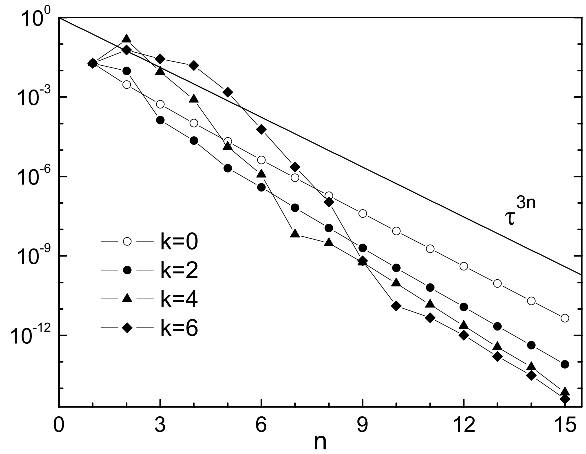

Figure 4. Convergence accelerated by more zeta functions. The errors of $\gamma_f^{(n)}$ and their weighted averages (see equations (55) and (56)) are plotted at $l = 1$.

We expect that this treatment can eliminate the possible pole(s) of $\zeta_0^{-1}(z, 0)$ located within the circle $|z| = z_c$. When l is sufficiently large, e.g. $l \geqslant 3$, we find the numerical evidence of such poles, namely, $\gamma^{(n)}$ converges slower than τ^{3ln}. Actually, the numerical calculation indicates that this treatment is helpful even when $\zeta_0^{-1}(z, 0)$ has no such pole: $\overline{\gamma}_{k,n}$ behaves as an asymptotic series, i.e. when n is fixed there is an optimal cutoff of k, which increases with n (see figure 4). Based on the numerical results listed in tables 1 and 2, we conclude that $\gamma_f \approx 0.123\,975\,598\,803\,35$.

5. Summary and discussion

In this paper we have studied the evolution operator method to improve the cycle expansion of the Lyapunov exponent of PRM with quenched disorder. The main result is that the convergence of the cycle expansion is closely related to the mixing of vector under the action

Table 1. Lyapunov exponent of the random Fibonacci sequence calculated with five fundamental matrices. n denotes the maximal cycle length and k denotes the number of additional zeta functions (see equations (55) and (56)).

n	$k = 0$	$k = 1$	$k = 2$	$k = 3$
1	0.1236	0.1236	0.1236	0.1236
2	0.1238	0.1237	0.1238	0.1238
3	0.123 98	0.123 977	0.1240	0.1240
4	0.123 974	0.123 9750	0.123 974	0.123 973
5	0.123 9756	0.123 975 57	0.123 975 64	0.123 9756
6	0.123 975 596	0.123 975 596	0.123 975 5981	0.123 975 5980
7	0.123 975 598 89	0.123 975 5986	0.123 975 598 81	0.123 975 598 79
8	0.123 975 598 79	0.123 975 598 79	0.123 975 598 8031	0.123 975 598 8032
9	0.123 975 598 803	0.123 975 598 802	0.123 975 598 803 35	0.123 975 598 803 35
10	0.123 975 598 8033	0.123 975 598 803	0.123 975 598 803 35	0.123 975 598 803 35

Table 2. As table 1, with seven fundamental matrices.

n	$k = 0$	$k = 1$	$k = 2$	$k = 3$
1	0.129	0.129	0.129	0.129
2	0.1237	0.124	0.1237	0.1238
3	0.123 99	0.123 99	0.123 98	0.123 979
4	0.123 974	0.123 977	0.123 9753	0.123 9754
5	0.123 9756	0.123 9757	0.123 9756	0.123 9756
6	0.123 975 594	0.123 9756	0.123 975 5987	0.123 975 5987
7	0.123 975 599	0.123 975 599	0.123 975 598 804	0.123 975 598 803 36
8	0.123 975 5987	0.123 975 598 84	0.123 975 598 803 33	0.123 975 598 803 35

of random matrices. Based on this fact, an accelerated algorithm for the cycle expansion is proposed, which greatly enhances the numerical efficiency since all exponentially converging terms are naturally eliminated. This method can be readily generalized to the case where the random matrices are generated from a Markovian model. To guarantee the hyperbolic condition, we restrict ourselves to positive matrices, or, in representation-independent words, we require that there is a sector in the vector space that strictly shrinks to its interior when transformed by the random matrices. This restriction means that when being tried to apply to some applications of PRM, e.g. the wave localization in random media, our method should be improved further.

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